ON A TRANSFORMATION OF THE WEIGHTED COMPOUND POISSON PROCESS

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1. COMPOUND POISSON VARIABLES AND WEIGHTED COMPOUND POISSON VARIABLES

We are using the following terminology—essentially following Feller:

a) *Compound Poisson Variable*

This is a random variable $Y = \sum_{j=0}^{N} X_j$

where $X_1, X_2, \ldots X_n, \ldots$ independent, identically distributed ($X_0 = 0$) and N a Poisson counting variable

hence

$G(x) = P(Y = x) = \sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} F^{*k}(X)$ where F stands for the

(common) distribution function of the X_j with $j \neq 0$ or in the language of characteristic functions

$$\psi(u) = \int e^{iux}\, dG(x) = e^{\lambda[\chi(u)-1]} \text{ with } \chi(u) = \int e^{iux}\, dF(x)$$

b) *Weighted Compound Poisson Variable*

This is a random variable Z obtained from a class of Compound Poisson Variables by weighting over λ with a weight function $S(\lambda)$

$$Z = \sum_{j=0}^{N} X_j \qquad \text{where } P(N = k) = \int_0^{\infty} e^{-\lambda} \frac{\lambda^k}{k!}\, dS(\lambda) = p_k$$

hence

$$H(x) = P(Z = x) = \sum_{k=0}^{\infty} p_k F^{*k}(x)$$

or in the language of characteristic functions

$$\varphi(u) = \int e^{iux}dH(x) = \int_0^\infty e^{\lambda[\chi(u)-1]}\,dS(\lambda) = \sum_{k=0}^\infty p_k[\chi(u)]^k$$

2. Description of the Transformation

Let $[Z(t)\,;\,t \geqslant 0]$ be a homogeneous Weighted Compound Poisson Process. The characteristic function at the time epoch t reads then

$$\varphi_t(u) = \int e^{\lambda t[\chi(u)-1]}\,dS(\lambda)$$

It is most remarkable that in many instances $\varphi_t(u)$ can be represented as a (non weighted) Compound Poisson Variable. Our main result is given as a theorem.

Theorem: The transformation

$$\int e^{\lambda t[\chi(u)-1]}\,dS(\lambda) = e^{\mu(t)[\tilde{\chi}_t(u)-1]} \quad \text{for all } t \geqslant 0$$

with $\mu(t) \geqslant 0$

$\tilde{\chi}_t(u)$ some characteristic function

is possible if and only if the weight function $S(\lambda)$ is infinitely divisible.

The stimulus to the discovery of this theorem—of importance in the collective risk theory—was provided by the Filip Lundberg Colloquium 1968 in Stockholm. In particular we refer to the paper of Thyrion. The special case where $S(\lambda)$ is the Γ-distribution was already obtained by Ammeter.

May we state that we can prove the necessity of the above condition only under the additional assumption that the mean $\int xdF(x)$ is finite, but we believe that the result even holds without this condition.

3. Terminology and Tools

a) *Infinitely Divisible Distributions*

$G(x)$ with characteristic function $\psi(u) = \int e^{iux}dG(x)$ is called *infinitely divisible* if any root $\psi^{1/n}(u)$ also represents a characteristic function. This means that the random variable belonging to $G(x)$ can be represented as the sum of any finite number of identically distributed independent summands.

b) *Theorem of Feller (page 271)*

Let $G(x)$ be infinitely divisible and concentrated on the non negative integers

Then $G(x)$ is necessarily Compound Poisson; i.e.

$\psi(u) = \int e^{iux} dG(x) = e^{\rho[\gamma(u)-1]}$ with $\rho \geqslant 0$
$\gamma(u)$ some characteristic function

4. Proof that the condition is necessary

We assume that $\int x dF(x)$ finite

Lemma: Let $\chi(u) = \int e^{iux} dF(x)$ and $\int x dF(x) = \alpha$ finite then $\lim_{n \to \infty} n[\chi(u/n) - 1] = iu\alpha$

Proof: $\lim_{n\to\infty} [\chi(u/n)]^n = e^{iu\alpha}$ by the law of large numbers, hence $\lim_{n\to\infty} n \log \chi(u/n) = iu\alpha$

Observe that

$$n \log \chi(u/n) = n[\chi(u/n) - 1 + o(\chi(u/n) - 1)]$$
$$| \chi(u/n) - 1 | \leqslant C \,.\, u/n \qquad C = \int | x | dF(x)$$

from which we conclude the lemma.

Theorem: Let $\varphi_t(u) = \int e^{\lambda t[\chi(u)-1]} dS(\lambda) = e^{\mu(t)[\tilde{\chi}_t(u)-1]}$ for all $t \geqslant 0$. Then $S(\lambda)$ must be infinitely divisible.

Proof: $\varphi_t(u)$ (as Compound Poisson characteristic function) is infinitely divisible and so

$$\varphi_n(u/n) = \int e^{\lambda n[\chi(u/n)-1]} dS(\lambda)$$

$\lim_{n\to\infty} \varphi_n(u/n) = \int e^{iu\alpha\lambda} dS(\lambda)$ by the lemma (dominated convergence)

hence

$g(u) = \int e^{iu\lambda} dS(\lambda)$ is an infinitely divisible characteristic function as limit of infinitely divisible characteristic functions.

5. Proof that the condition is sufficient

Theorem: Let $S(\lambda)$ be infinitely divisible

Then $\varphi_t(u) = \int e^{\lambda t[\chi(u)-1]}\, dS(\lambda) = e^{\mu(t)[\tilde{\chi}_t(u)-1]}$

for all $t \geqslant 0$
with $\mu(t) \geqslant 0$
$\bar{\chi}_t(u)$ characteristic function

Proof: We write $p_k(t)$ for $\int e^{-\lambda t} \frac{(\lambda t)^k}{k!}\, dS(\lambda)$

Let $\sigma(u) = \int e^{iu\lambda}\, dS(\lambda)$

$$\delta_t(u) = \sum_{k=0}^{\infty} e^{iuk} p_k(t) = \sum_{k=0}^{\infty} e^{iuk} \int e^{-\lambda t} \frac{(\lambda t)^k}{k!}\, dS(\lambda) =$$

$$= \int e^{\lambda t(e^{iu}-1)}\, dS(\lambda)$$

hence
$$\delta_t(u) = \sigma\left[\frac{t(e^{iu}-1)}{i}\right]$$

From $\sigma(u)$ infinitely divisible follows with a suitable distribution function $S_n(\lambda)$
$\sigma^{1/n}(u) = \int e^{iu\lambda}\, dS_n(\lambda)$ for arbitrary (even complex) u whenever the integral exists

and
$$\delta_t^{1/n}(u) = \sigma^{1/n}\left(\frac{t(e^{iu}-1)}{i}\right) = \int e^{\lambda t(e^{iu}-1)}\, dS_n(\lambda)$$

which again is a characteristic function; hence $\delta_t(u)$ infinitely divisible.

It follows from the above mentioned theorem of Feller that

$$\delta_t(u) = e^{\rho(t)[\gamma_t(u)-1]} \text{ with } \gamma_t(u) = \Sigma\, e^{iuk} g_k(t)$$

We obtain finally

$$\varphi_t(u) = \sum_{k=0}^{\infty} p_k(t)\, [\chi(u)]^k = \delta_t\left(\frac{\ln \chi(u)}{i}\right) = \exp\left\{\rho(t)\left[\gamma_t\left(\frac{\ln \chi(u)}{i}\right) - 1\right]\right\}$$

As $\bar{\chi}_t(u) = \sum_{R=0}^{\infty} g_k(t)\, [\chi(u)]^k = \gamma_t\left(\frac{\ln \chi(u)}{i}\right)$ is a characteristic function this concludes the proof.

6. Bibliography

[1] W. Feller: An Introduction to Probability Theory and its Applications Volume 1, Wiley, New York/London 1957 (second edition).
[2] P. Thyrion: Théorie collective du risque, Filip Lundberg Colloquium 1968.
[3] H. Ammeter: A generalization of the collective theory of risk in regard to fluctuating basic probabilities Skandinavisk Aktuarietdskrift, 1948 (volume 31).